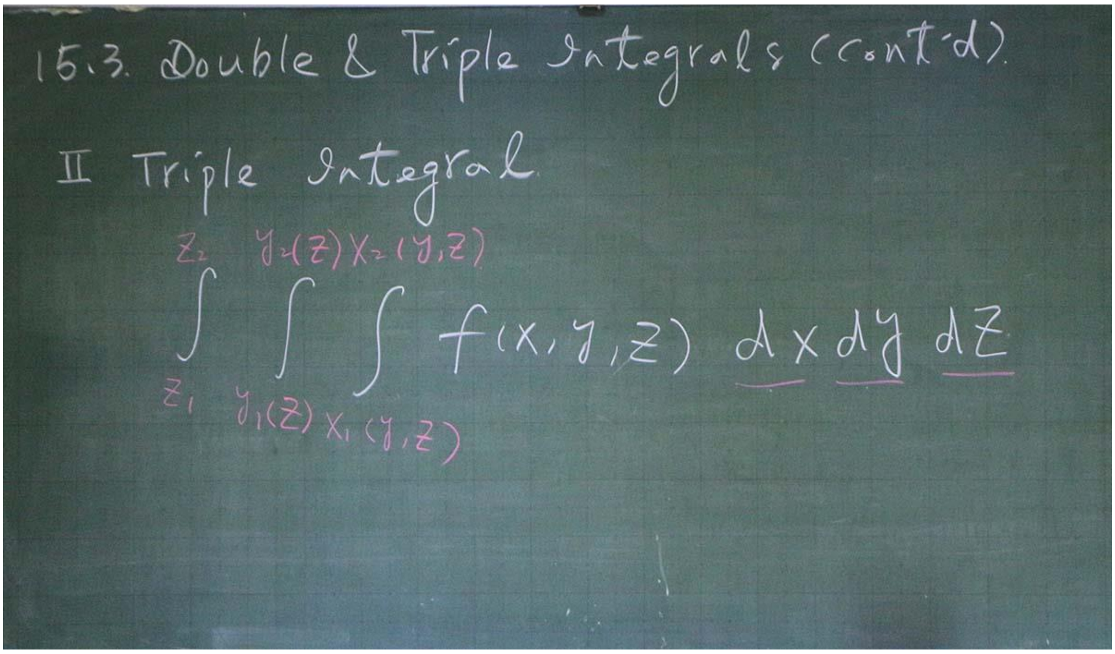
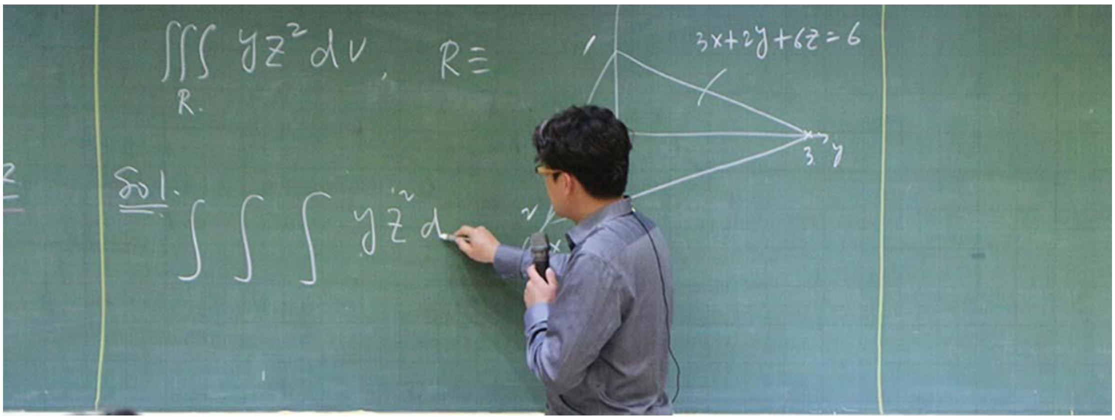
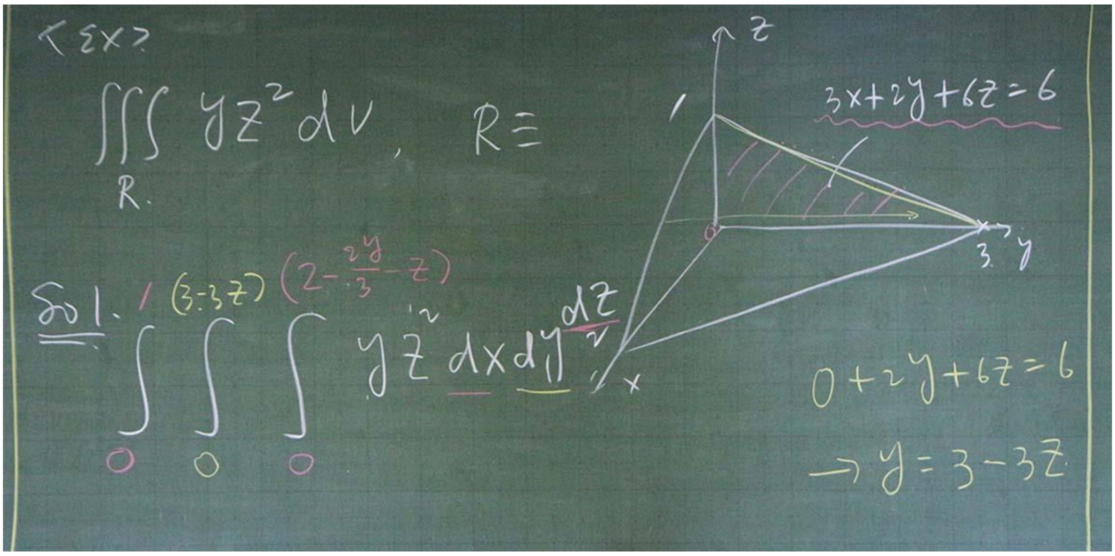
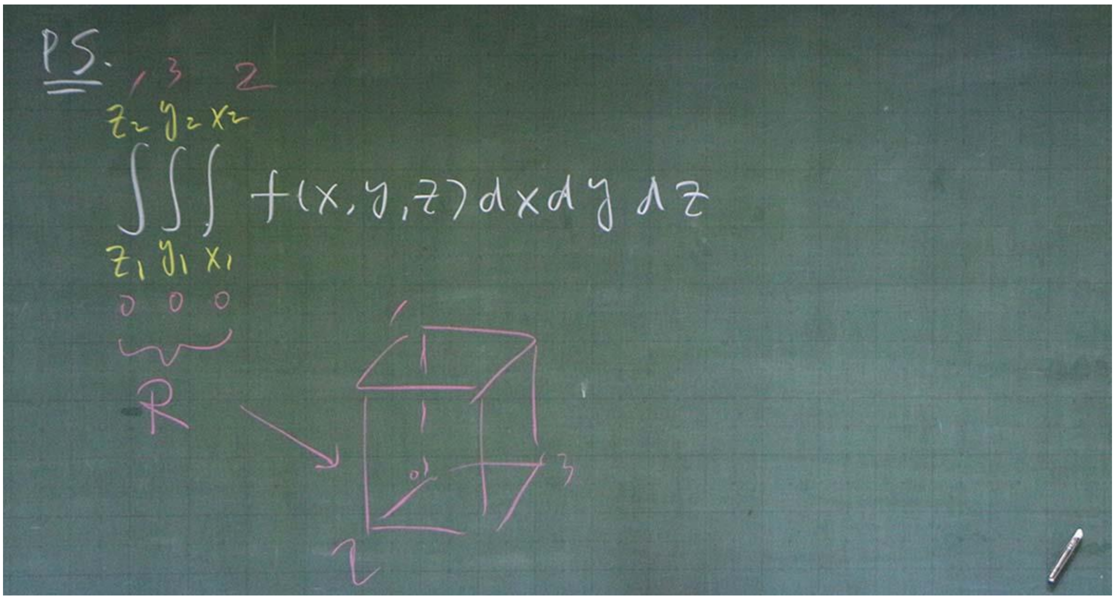
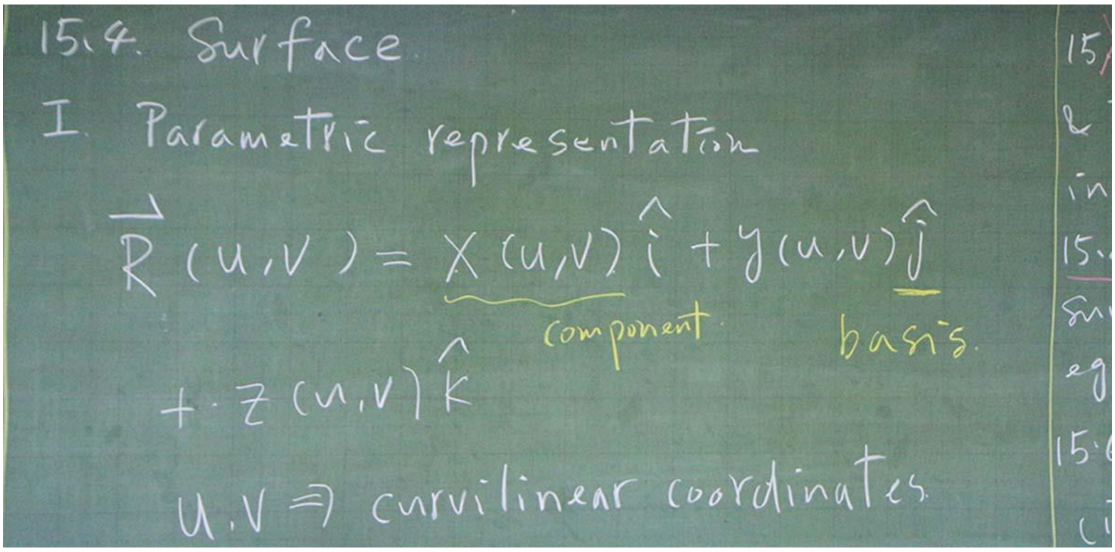

15.3. Double & Triple Integrals (cont'd)
II Triple Integral
$\int_{z_1}^{z_2}\int_{y_1(z)}^{y_2(z)}\int_{x_1(y,z)}^{x_2(y,z)} f(x,y,z)\,dx\,dy\,dz$


$\iiint_R yz^2\,dV$, R =
1
3x+2y+6z=6
3 y
Sol. $\int\int\int yz^2\,d$

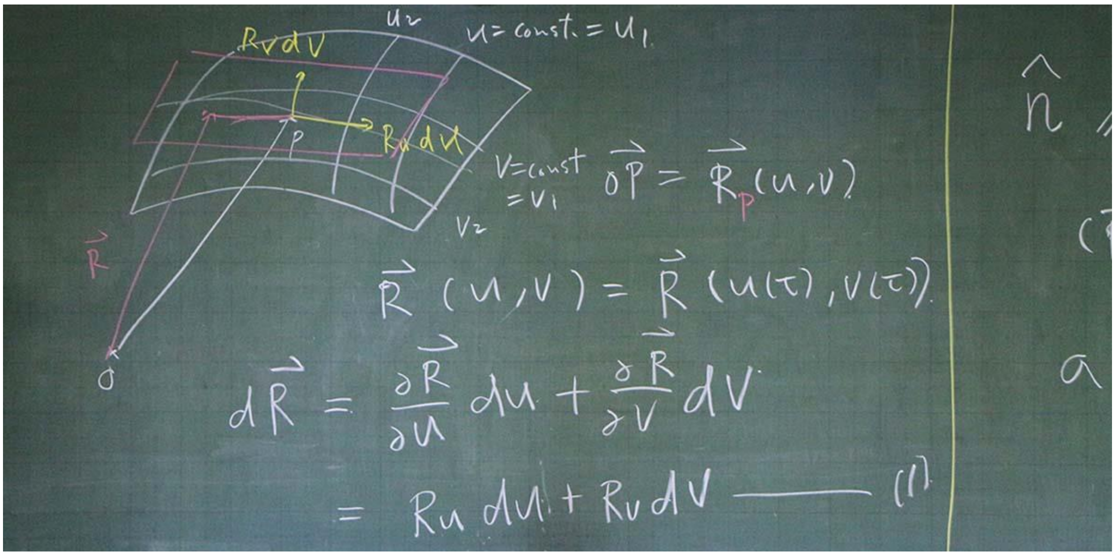

<Ex>
$\iiint_R yz^2\,dV$, R =
z
1
3x+2y+6z=6
3 y
x
Sol. $\int_0^1\int_0^{(3-3z)}\int_0^{\left(2-\frac{2y}{3}-z\right)} yz^2\,dx\,dy\,dz$
$0+2y+6z=6$
$\rightarrow y=3-3z$

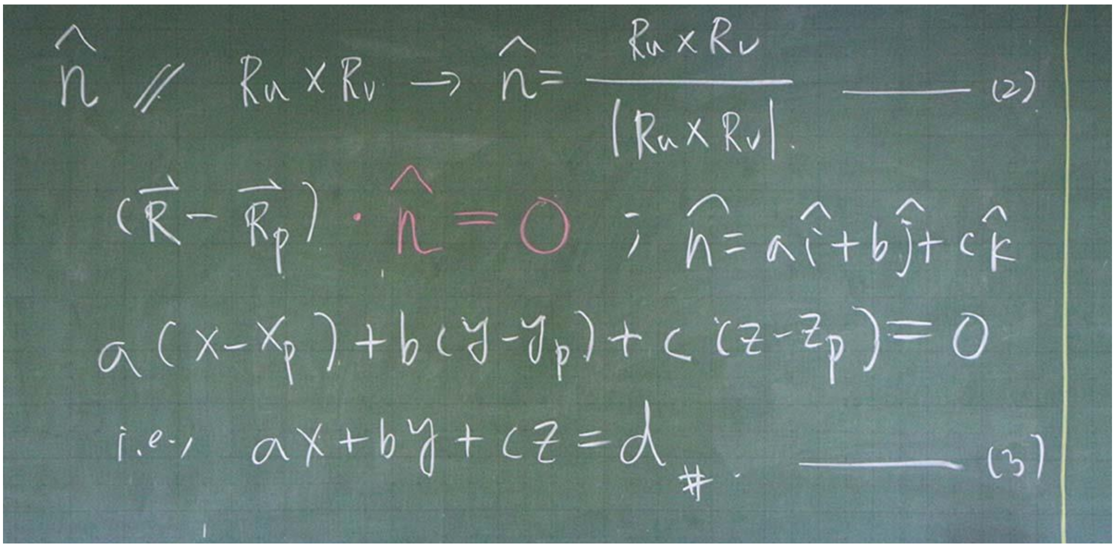

PS.
1 3 2
z2 y2 x2
∫∫∫ f(x,y,z) dx dy dz
z1 y1 x1
0 0 0
R
1
3
2


∫∫∫ f(x,y,z) dx dy dz
z1 y1 x1
R
R(u,v) = X
î + y(u,v) ĵ
basis
+ z(u,v) k̂
u.v ⇒
coordinates


15.4. Surface
I. Parametric representation
R(u,v) = X(u,v) î + y(u,v) ĵ
component
basis
+ z(u,v) k̂
u.v ⇒ curvilinear coordinates


z1 y1 x1
R
1
2
3


Rv dv
u2
u = const. = u1
P
Ru du
v=const
=v1
$\vec{OP} = \vec{R}_p(u,v)$
v2
$\vec{R}$
O
$\vec{R}(u,v) = \vec{R}(u(\tau), v(\tau))$
$d\vec{R} = \frac{\partial \vec{R}}{\partial u} du + \frac{\partial \vec{R}}{\partial v} dv$
$= R_u du + R_v dv$ —— (1)


$\hat{n} \parallel R_u \times R_v \rightarrow \hat{n} = \frac{R_u \times R_v}{|R_u \times R_v|}$ —— (2)
$(\vec{R} - \vec{R}_p) \cdot \hat{n} = 0$; $\hat{n} = a\hat{i} + b\hat{j} + c\hat{k}$
$a(x - x_p) + b(y - y_p) + c(z - z_p) = 0$
i.e., $ax + by + cz = d$ # —— (3)

II.

$f(x, y, z) = 0$, find the tangent plane at P

TF of f at $P \rightarrow f(x, y, z)$

$$= \underbrace{f(x_p, y_p, z_p)}_{\downarrow\ 0} + \Big[f_x (x - x_p) + f_y (y - y_p)$$

$$+ f_z (z - z_p) \Big] + \underbrace{\cdots}_{\text{high-order terms!}}$$

$x \rightarrow x_p$

$$\underbrace{f(x, y, z)}_{\text{tangent plane}} = \underset{\updownarrow\ a}{f_x} (x - x_p) + \underset{\updownarrow\ b}{f_y} (y - y_p) + \underset{\updownarrow\ c}{f_z} (z - z_p)$$

$$\hat{n} = \frac{f_x \hat{i} + f_y \hat{j} + f_z \hat{k}}{(f_x^2 + f_y^2 + f_z^2)^{1/2}} \quad \#$$

15.~~3~~ Dou
& Triple
integra
15.~~4~~ & 15.
Surface (
egral)
15.6 Volu
(integral

15.5 Surface Integral

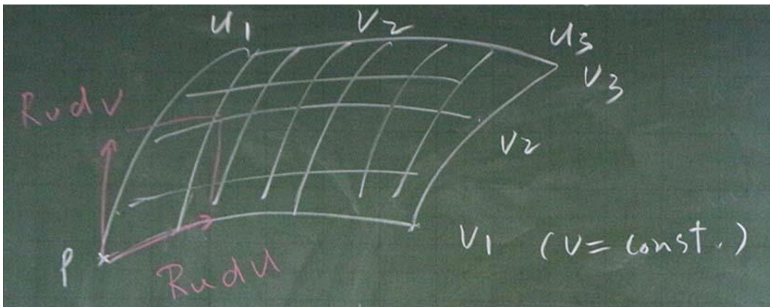
I.

$$\iint f(x, y, z, t)\, dA$$

$$dA = dx\,dy \ ; \ dy\,dx$$ "Cartesian coor.

$dA = ?$

Ru

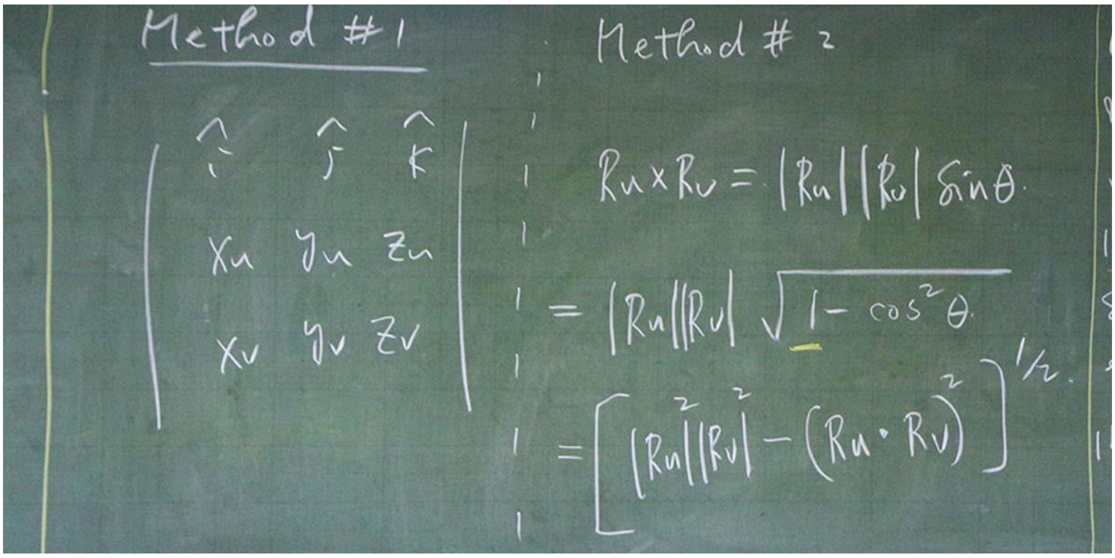


$$dA = (R_u \times R_v)\, du\, dv \quad \#$$

$$\vec{R}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}.$$

$$R_u = x_u\,\hat{i} + y_u\,\hat{j} + z_u\,\hat{k}$$

$$R_v = x_v\,\hat{i} + y_v\,\hat{j} + z_v\,\hat{k}$$


Method #1
Method #2
$R_u \times R_v = |R_u||R_v| \sin\theta$
$= |R_u||R_v| \sqrt{1-\cos^2\theta}$
$= \left[|R_u|^2|R_v|^2 - (R_u \cdot R_v)^2\right]^{1/2}$

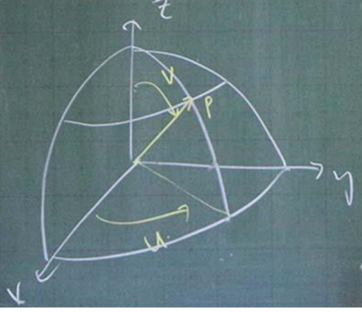

integrals
Surface (integral)
15.6 Volume (integral)


Triple
integrals
Surface (integral)
15.6 Volume (integral)

<EX> Find the surface of a sphere w/ the radius a.

Sol

$$dA = |R_u \times R_v| \, du \, dv$$



$$x = a \sin v \cos u$$
$$y = a \sin v \sin u$$
$$z = a \cos v$$

u: longitude (經度)

v: latitude (緯度)

$$\vec{R} = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

$$\vec{R_u} = -a \sin v \sin u \,\hat{i} + a \sin v \cos u \,\hat{j}$$

$$\vec{R_v} = a \cos v \cos u \,\hat{i} + a \cos v \sin u \,\hat{j} - a \sin v \,\hat{k}$$

$$|\vec{R_u} \times \vec{R_v}| = \left[|\vec{R_u}|^2 |\vec{R_v}|^2 - (\vec{R_u} \cdot \vec{R_v})^2 \right]^{1/2}$$

$$= a^2 \left[(\sin^2 v \sin^2 u + \sin^2 v \cos^2 u)(\cos^2 v \cos^2 u + \cos^2 v \sin^2 u + \sin^2 v) - 0 \right]^{1/2}$$

$$= a^2\left[(\sin^2 v)\underbrace{(\cos^2 v + \sin^2 v)}_{1}\right]^{1/2}$$

$$= a^2 \sin v \quad \#$$

$$8 \times \int_{\pi/2}^{\pi/2}\int_{0}^{\pi/2} a^2 \sin v \, du \, dV = 4\pi a^2 \quad \#$$

<u>Notes</u> (x2).

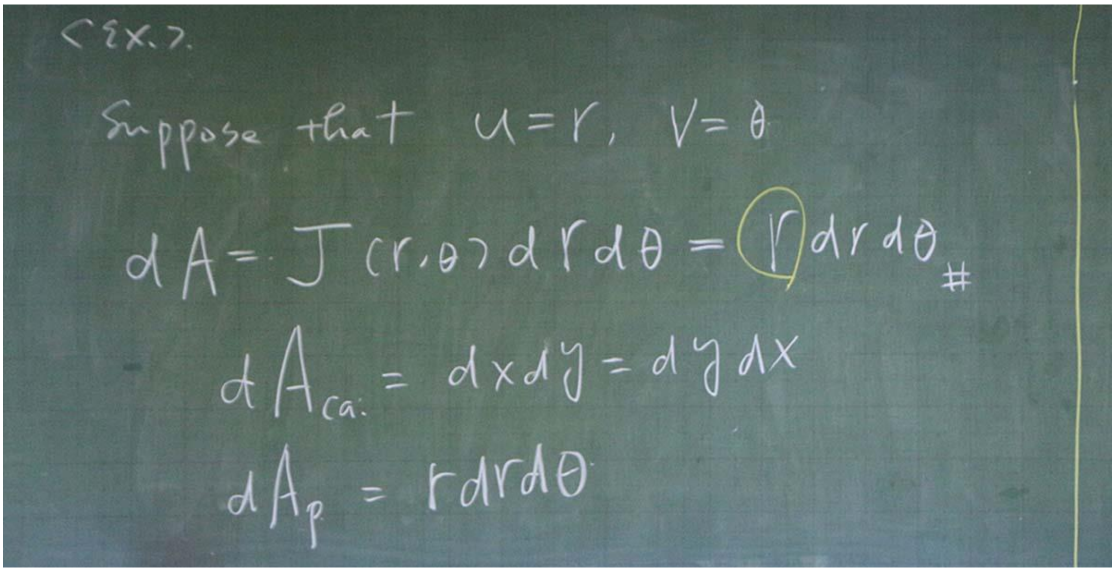
N1. For 2D cases (i.e., $Z = \text{const.}$)

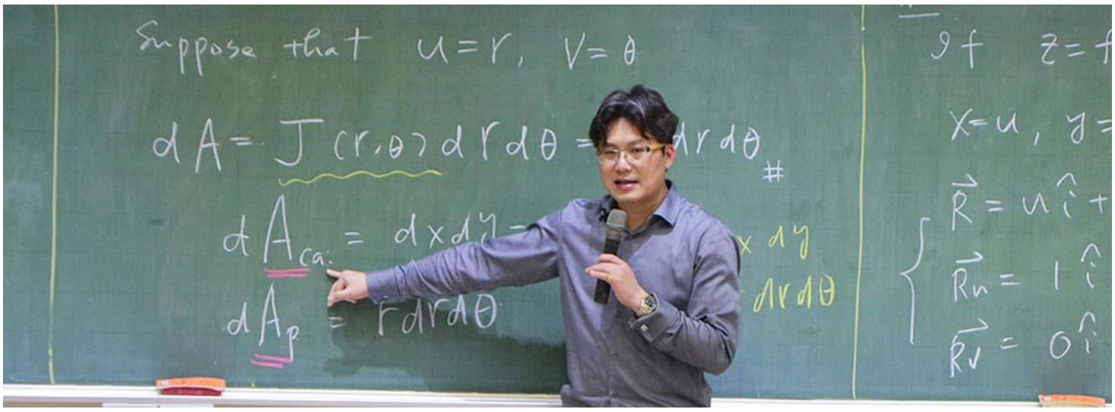
$$\vec{R_u} \times \vec{R_v} = \Big[\underbrace{(x_u^2 + y_u^2)}_{|R_u|^2}\underbrace{(x_v^2 + y_v^2)}_{|R_v|^2} - (x_u x_v + y_u y_v)^2\Big]^{1/2}$$

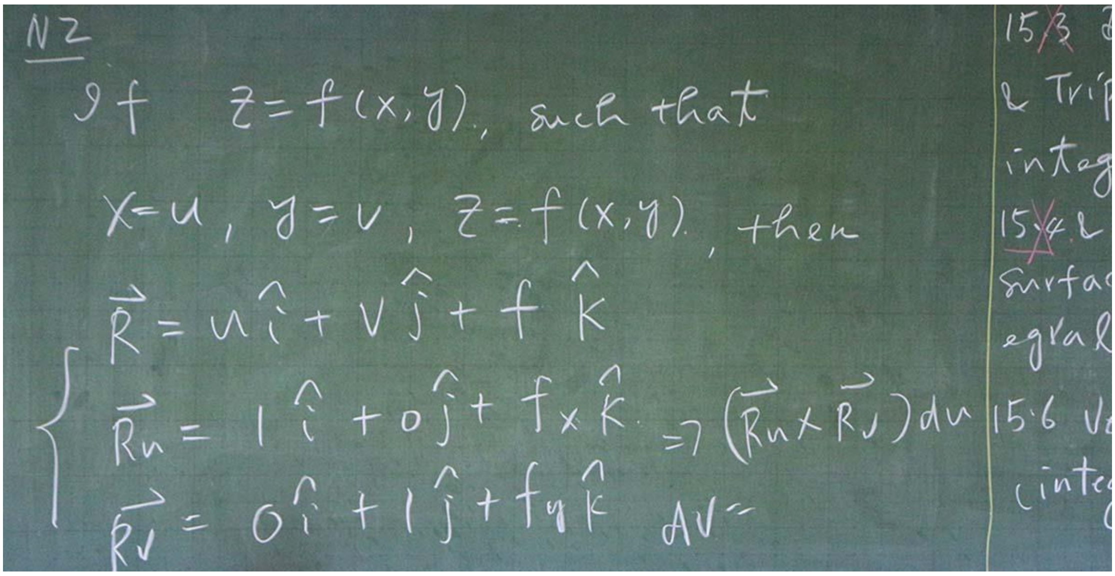
$$= |x_u y_v - x_v y_u| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}$$

$$= J(u, v)$$

Namely, $|\vec{R_u} \times \vec{R_v}|\, du\, dV = J(u, v)\, du\, dv$


<Ex.>
Suppose that u = r, V = θ
dA = J(r,θ) dr dθ = r dr dθ #
dA_ca = dx dy = dy dx
dA_p = r dr dθ


Suppose that u = r, V = θ
dA = J(r,θ) dr dθ
dA_ca = dx dy
dA_p = r dr dθ


N2
If z = f(x,y), such that
x = u, y = v, z = f(x,y), then
R = u î + v ĵ + f k̂
Ru = 1 î + 0 ĵ + fx k̂ ⇒ (Ru × Rv) du
Rv = 0 î + 1 ĵ + fy k̂ dv
15.3
& Triple
integ
15.4
Surfac
egral
15.6 V
(integ

15. Volumn & Volumn Integral.

I. Volumn integral

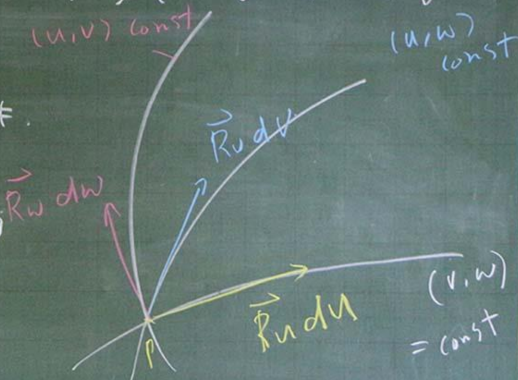
$$\iiint f(x, y, z, t)\, \underline{\underline{dV}} \quad ?$$

For Cartesian coordinates,

$$dV = dx\,dy\,dz.$$

$$\vec{R}(u,v,w) = x(u,v,w)\,\hat{i} + y(u,v,w)\,\hat{j} + z(u,v,w)\,\hat{k} \quad \#$$

$$dV = \left|\vec{R}_u \times \vec{R}_v \cdot \vec{R}_w\right| du\,dv\,dw$$



$$= \left\| \begin{matrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{matrix} \right\| du\,dv\,dw$$

absolute value

determinant

$$= \left\| \begin{matrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{matrix} \right\|$$

P.S. $A \to A^{T}$ (T: transpose).

$$\det A = \det A^{T}.$$

$$= J(u,v,w)\,du\,dv\,dw = dV \quad \#$$

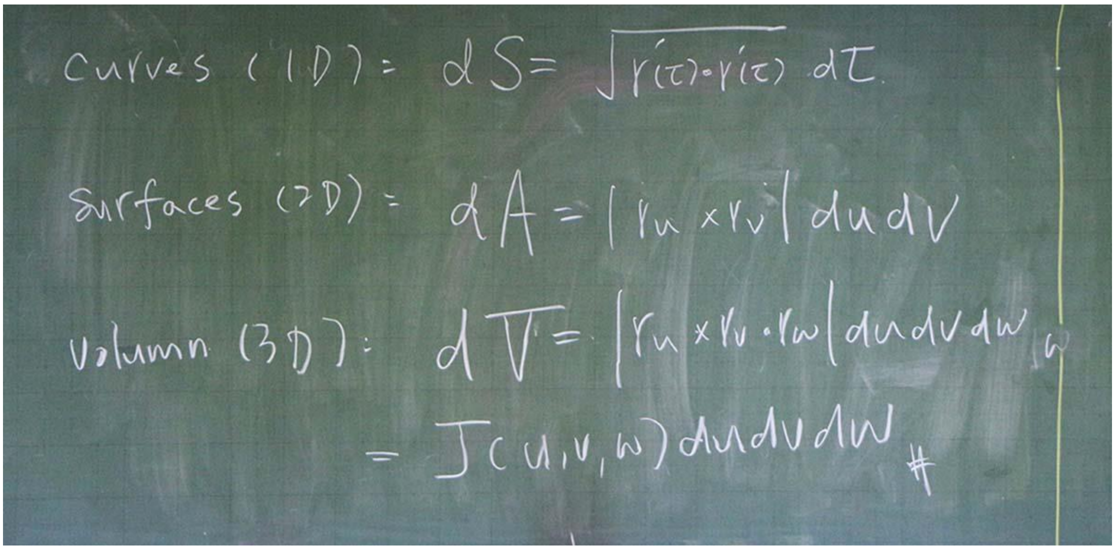
* cylindrical coordinates

$$dV = J(r,\theta,z)\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz$$

* spherical coordinate

$$dV = J(\rho, \underset{\text{lat.}}{\underline{\underline{\phi}}}, \underset{\text{long.}}{\underline{\theta}})\,d\rho\,d\phi\,d\theta$$

$$= \rho^{2}\,|\sin\phi|\,d\rho\,d\phi\,d\theta$$


Curves (1D) = $dS = \sqrt{r'(t) \cdot r'(t)}\, dt$.
Surfaces (2D) = $dA = |r_u \times r_v|\, du\, dv$
Volumn (3D): $dV = |r_u \times r_v \cdot r_w|\, du\, dv\, dw$
$= J(u,v,w)\, du\, dv\, dw$ #